Preprint typeset in JHEP style - HYPER VERSION

Three-dimensional topologically gauged $\mathcal{N}=6$ ABJM type theories

Xiaoyong Chu and Bengt E.W. Nilsson

Fundamental Physics
Chalmers University of Technology
SE-412 96 Göteborg, Sweden

xiaoyong@student.chalmers.se, tfebn@chalmers.se

Abstract: In this paper we construct the $\mathcal{N}=6$ conformal supergravity in three dimensions from a set of Chern-Simons-like terms one for each of the graviton, gravitino, and R-symmetry gauge field and then couple this theory to the $\mathcal{N}=6$ superconformal ABJM theory. In a first step part of the coupled Lagrangian for this topologically gauged ABJM theory is derived by demanding that all terms of third and second order in covariant derivatives cancel in the supersymmtry variation of the Lagrangian. To achieve this the transformation rules of the two separate sectors must be augmented by new terms. In a second step we analyze all terms in δL that are of first order in covariant derivatives. The cancelation of these terms require additional terms in the transformation rules as well as a number of new terms in the Lagrangian. As a final step we check that all remaining terms in δL which are bilinear in fermions cancel which means that the presented Lagrangian and transformation rules constitute the complete answer. In particular we find in the last step new terms in the scalar potential containing either one or no structure constant. The non-derivative higher fermion terms in δL that have not yet been completely analyzed are briefly discussed.

Keywords: String theory, M-theory, Branes, Chern-Simons theory.

1. Introduction

Conformal field theories in three dimensions have recently experienced a number of interesting developments. The perhaps most unexpected and profound results are the actual construction of a seemingly unique three-dimensional maximally ($\mathcal{N} = 8$) superconformal theory by Bagger and Lambert, and by Gustavsson (BLG) [1, 2, 3, 4], along with its $\mathcal{N} = 6$ variant (ABJM) by the authors of [5, 6].

In [7] an attempt was made to couple the $\mathcal{N} = 8$ BLG theory to conformal supergravity. After presenting a detailed derivation of pure $\mathcal{N} = 8$ conformal supergravity, this work went on to take a first step towards the construction of a Lagrangian describing the coupling of this theory to the BLG theory. By checking that the supersymmetry variation of the coupled Lagrangian vanishes for terms of third and second order in covariant derivatives a set of coupling terms were obtained. This procedure also led to a number of new terms (as compared to the uncoupled theories) in the supersymmetry variation of the two spin one gauge fields that enter these two theories, namely the ones corresponding to the $SO(8)$ R-symmetry and BLG gauge symmetry. However, the rigidity of the BLG theory seems at this point to prevent a straightforward continuation of this construction. For a brief discussion why such topologically coupled theories might be of interest in connection with M-theory and AdS/CFT, see the introductory section of [7].

In this paper we demonstrate that these results can be rather easily obtained also for the $\mathcal{N} = 6$ case. In fact, we will also show that one can carry this construction, without meeting any serious obstacles, all the way giving in the end the entire Lagrangian and transformation rules. As explained in the Conclusions, however, the proof of supersymmetry is not yet completed since some of the non-derivative higher fermion terms in the variation of the Lagrangian remain to be checked. The paper is organized as follows. In section two we summarize our results on the Lagrangian and transformation rules for the coupled theory. We start the derivation of these results in section three by constructing the $\mathcal{N} = 6$ conformal supergravity theory, and then go on in section four to review the ABJM matter sector. With these two ingredients at hand, in section five we take the first step in the process of coupling these two theories by carrying out the same analysis as in [7] where it was done in detail in the $\mathcal{N} = 8$ case. This step amounts to checking the cancelation of all terms with two covariant derivatives in δL. To get this to work we are forced to add new terms to the transformation rules of the R-symmetry and ABJM gauge fields. The following section, section six, contains the second step where all terms in δL containing one covariant derivative are checked and seen to cancel. This step requires a number of new terms in both the Lagrangian and in the transformation rules, in particular we find a $U(1)$ gauge field to play a special role. The terms in δL bilinear in fermions and without derivatives are then discussed in section seven and shown to cancel. This step brings in new six-scalar terms in the potential which have either one or no

structure constant. The theory obtained at this point can be shown to be the full theory. However, the proof of supersymmetry is yet not completed in all details. The terms in δL that have not been checked so far are discussed in a concluding section. These terms are all without derivatives and contain more than two fermions.

2. The complete Lagrangian and transformation rules of topologically gauged $\mathcal{N}=6$ ABJM theories: a summary

In this section we state the final result of this paper, that is, the complete Lagrangian and supersymmetry transformation rules. The invariance under the $\mathcal{N}=6$ supersymmetries is checked in the following sections for all terms in δL containing covariant derivatives, as well as for all non-derivative terms that are bilinear in fermions (including the supersymmetry parameter). At this point in the construction we are able to conclude that the results obtained constitute the complete theory.

2.1 The ansatz for the Lagrangian and transformation rules

We find that the Lagrangian is given by (with $A=\pm\sqrt{2}$)

$$
\begin{aligned}
L = {} & L^{conf}_{sugra} + L^{cov}_{ABJM} + \tfrac{1}{2}\epsilon^{\mu\nu\rho}C_\mu\partial_\nu C_\rho && (2.1)\\
& + iAe\bar{\chi}^{BA}_\mu\gamma^\nu\gamma^\mu\Psi_{Aa}(\tilde{D}_\nu\bar{Z}^a_B - \tfrac{i}{2}A\bar{\chi}_{\nu BC}\Psi^{Ca}) + c.c. && (2.2)\\
& + i\epsilon^{\mu\nu\rho}(\bar{\chi}^{AC}_\mu\chi_{\nu BC})Z^B_a\tilde{D}_\rho\bar{Z}^a_A + c.c. && (2.3)\\
& - iA(\bar{f}^{\mu AB}\gamma_\mu\Psi_{Aa}\bar{Z}^a_B + \bar{f}^\mu_{AB}\gamma_\mu\Psi^{Aa}Z^B_a) && (2.4)\\
& - \tfrac{e}{8}\tilde{R}|Z|^2 + \frac{i}{2}|Z|^2\bar{f}^\mu_{AB}\chi^{AB}_\mu && (2.5)\\
& + 2ieAf^{ab}{}_{cd}(\bar{\chi}_{\mu AB}\gamma^\mu\Psi^{d[B})Z^{D]}_a Z^A_b\bar{Z}^c_D + c.c. && (2.6)\\
& - i\epsilon^{\mu\nu\rho}(\bar{\chi}_{\mu AB}\gamma_\nu\chi^{CD}_\rho)(Z^A_a Z^B_b\bar{Z}^c_C\bar{Z}^d_D)f^{ab}{}_{cd} && \\
& + \frac{i}{4}\epsilon^{\mu\nu\rho}(\bar{\chi}_{\mu AB}\gamma_\nu\chi^{AB}_\rho)(Z^C_a Z^D_b\bar{Z}^c_C\bar{Z}^d_D)f^{ab}{}_{cd} && (2.7)\\
& - \frac{i}{16}e\epsilon^{ABCD}(\bar{\Psi}_{Aa}\Psi_{Bb})\bar{Z}^a_C\bar{Z}^b_D + c.c. && \\
& + \frac{i}{16}e(\bar{\Psi}_{Db}\Psi^{Db})|Z|^2 - \frac{i}{4}e(\bar{\Psi}_{Db}\Psi^{Bb})\bar{Z}^a_B Z^D_a && \\
& + \frac{i}{8}e(\bar{\Psi}_{Db}\Psi^{Da})\bar{Z}^b_B Z^B_a + \frac{3i}{8}e(\bar{\Psi}_{Db}\Psi^{Ba})\bar{Z}^b_B Z^D_a && (2.8)\\
& - \frac{i}{16}eA(\bar{\chi}_{\mu AB}\gamma^\mu\Psi^{Bb})|Z|^2 Z^A_b - \frac{i}{4}eA(\bar{\chi}_{\mu AB}\gamma^\mu\Psi^{Db})Z^A_a Z^B_b\bar{Z}^a_D + c.c && (2.9)\\
& - \frac{i}{4}\epsilon^{\mu\nu\rho}(\bar{\chi}_{\nu AB}\gamma_\rho\chi^{CD}_\mu)Z^A_a Z^B_b\bar{Z}^a_C\bar{Z}^b_D + \frac{i}{64}\epsilon^{\mu\nu\rho}(\bar{\chi}_{\nu AB}\gamma_\rho\chi^{AB}_\mu)|Z|^4 && (2.10)\\
& + \frac{1}{8}ef^{ab}{}_{cd}|Z|^2 Z^C_a Z^D_b\bar{Z}^c_C\bar{Z}^d_D + \frac{1}{2}ef^{ab}{}_{cd}Z^B_a Z^C_b Z^D_e\bar{Z}^e_B\bar{Z}^c_C\bar{Z}^d_D && (2.11)\\
& + \frac{5}{12\cdot 64}e|Z|^6 - \frac{1}{32}e|Z|^2 Z^A_b Z^C_a\bar{Z}^b_C\bar{Z}^a_A + \frac{1}{48}eZ^A_a Z^B_b Z^C_d\bar{Z}^b_A\bar{Z}^d_B\bar{Z}^a_C, && (2.12)
\end{aligned}
$$

where *c.c.* refers to complex conjugation of the expression on the line where it occurs.

This Lagrangian has some features that need to be clarified at this point[1]. The first one concerns the ABJM Dirac term that after gauging will be written in the self-conjugate way[2]

$$-\tfrac{ie}{2}\bar{\Psi}^{Aa}\gamma^{\mu}\tilde{D}_{\mu}\Psi_{Aa}-\tfrac{ie}{2}\bar{\Psi}_{Aa}\gamma^{\mu}\tilde{D}_{\mu}\Psi^{Aa}. \tag{2.13}$$

Secondly, the covariant derivative used here is defined by

$$\tilde{D}_{\mu}\psi^{Aa}=\partial_{\mu}\psi^{Aa}+\tfrac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\psi^{Aa}+B_{\mu B}^{A}\psi^{Ba}+\tilde{A}_{\mu b}^{a}\psi^{Ab}+qC_{\mu}\psi^{Aa}, \tag{2.14}$$

where attention should be paid to the presence of the last term. The Chern-Simons term for this abelian gauge field is written explicitly in the Lagrangian given above and the reason for giving the matter fields a charge q under this explicit $U(1)$ will become clear later when we explain how we obtain the topologically gauged ABJM Lagrangian. We will then also see that $q^2=\frac{1}{16}$.

The purpose of this paper is to show that the above Lagrangian is $\mathcal{N}=6$ supersymmetric. We have found that this is the case if the fields transform as follows:

$$\delta e_{\mu}{}^{\alpha}=i\bar{\epsilon}_{gAB}\gamma^{\alpha}\chi_{\mu}^{AB}, \tag{2.15}$$

$$\delta\chi_{\mu}^{AB}=\tilde{D}_{\mu}\epsilon_{g}^{AB}, \tag{2.16}$$

$$\begin{aligned}\delta B_{\mu}{}^{A}{}_{B}=&\,\frac{i}{e}(\bar{f}^{\nu AC}\gamma_{\mu}\gamma_{\nu}\epsilon_{gBC}-\bar{f}^{\nu}_{BC}\gamma_{\mu}\gamma_{\nu}\epsilon_{g}^{AC})\\&+\tfrac{i}{4}(\bar{\epsilon}_{BD}\gamma_{\mu}\Psi^{a(D}Z_{a}^{A)}-\bar{\epsilon}^{AD}\gamma_{\mu}\Psi_{a(D}\bar{Z}_{B)}^{a})\\&-\tfrac{i}{2}(\bar{\epsilon}_{g}^{AC}\chi_{\mu DC}Z_{a}^{D}\bar{Z}_{B}^{a}-\bar{\epsilon}_{gBC}\chi_{\mu}^{DC}Z_{a}^{A}\bar{Z}_{D}^{a})\\&+\tfrac{i}{8}\delta_{B}^{A}(\bar{\epsilon}_{g}^{EC}\chi_{\mu DC}-\bar{\epsilon}_{gDC}\chi_{\mu}^{EC})Z_{a}^{D}\bar{Z}_{E}^{a}\\&+\tfrac{i}{8}(\bar{\epsilon}_{g}^{AD}\chi_{\mu BD}-\bar{\epsilon}_{gBD}\chi_{\mu}^{AD})|Z|^{2},\end{aligned} \tag{2.17}$$

$$\delta Z_{a}^{A}=i\bar{\epsilon}^{AB}\Psi_{Ba}, \tag{2.18}$$

$$\begin{aligned}\delta\Psi_{Bd}=&\,\gamma^{\mu}\epsilon_{AB}(\tilde{D}_{\mu}Z_{d}^{A}-iA\bar{\chi}_{\mu}^{AD}\Psi_{Dd})\\&+f^{ab}{}_{cd}Z^{C}{}_{a}Z_{b}^{D}\bar{Z}_{B}^{c}\epsilon_{CD}-f^{ab}{}_{cd}Z_{a}^{A}Z_{b}^{C}\bar{Z}_{C}^{c}\epsilon_{AB}\\&+\frac{1}{4}Z_{c}^{C}Z_{d}^{D}\bar{Z}_{B}^{c}\epsilon_{CD}+\frac{1}{16}|Z|^{2}Z_{d}^{A}\epsilon_{AB},\end{aligned} \tag{2.19}$$

$$\begin{aligned}\delta\tilde{A}_{\mu}{}^{c}{}_{d}=&-i(\bar{\epsilon}_{AB}\gamma_{\mu}\Psi^{Aa}Z_{b}^{B}-\bar{\epsilon}^{AB}\gamma_{\mu}\Psi_{Ab}\bar{Z}_{B}^{a})f^{bc}{}_{ad}\\&-2i(\bar{\epsilon}_{g}^{AD}\chi_{\mu BD}-\bar{\epsilon}_{gBD}\chi_{\mu}^{AD})Z_{b}^{B}\bar{Z}_{A}^{a}f^{bc}{}_{ad},\end{aligned} \tag{2.20}$$

$$\begin{aligned}\delta C_{\mu}=&-iq(\bar{\epsilon}_{AB}\gamma_{\mu}\Psi^{Aa}Z_{a}^{B}-\bar{\epsilon}^{AB}\gamma_{\mu}\Psi_{Aa}\bar{Z}_{B}^{a})\\&-2iq(\bar{\epsilon}_{g}^{AD}\chi_{\mu BD}-\bar{\epsilon}_{gBD}\chi_{\mu}^{AD})Z_{a}^{B}\bar{Z}_{A}^{a},\end{aligned} \tag{2.21}$$

[1]For comments about the introduction of a dimensionless gravitational coupling constant and levels, see the concluding section.

[2]The SU(4) indices are used to keep track of complex conjugation while the bar indicates if the SO(2,1) spinor index has been raised or lowered with a charge conjugation matrix (which are never written out explicitly).

where $\epsilon_m^{AB} = A\epsilon_g^{AB} = \epsilon^{AB}, A = \pm\sqrt{2}$ and $q^2 = \frac{1}{16}$.

Finally we note that explicit covariant derivatives appear only in two terms in the Lagrangian, namely the supercurrent term and, on the following line in the Lagrangian above, the $\chi\chi ZDZ$ term. There is also an explicit covariant derivative in the transformation rules of the Rarita-Schwinger field and the ABJM fermion. In this context we note that in the latter case the derivative is made supercovariant by adding a second term giving the factor $(DZ - \chi\Psi)$. The same has to be done in the supercurrent term in the Lagrangian but with an extra factor of $\frac{1}{2}$. Note, however, that the other derivative term in L is not augmented with a similar term. We have checked that such a term, which would be cubic in χ, has zero coefficient. Thus all terms with more than two χ fields are in fact absorbed into the covariant derivatives and field strengths in the Lagrangian.

The demonstration of supersymmetry carried out in the following sections is divided into several steps starting with a construction of $\mathcal{N} = 6$ conformal supergravity. This is followed by adding on the ABJM theory and a stepwise incorporation of various subsets of the interaction terms given above as supersymmetry is checked for more and more terms in δL, organized in decreasing order in covariant derivatives.

3. Pure topological $\mathcal{N} = 8$ and $\mathcal{N} = 6$ supergravity in three dimensions

For $\mathcal{N} = 1$ a conformal and locally supersymmetric gravity theory in three dimensions consisting of two Chern-Simons like terms was shown to exist by Deser and Kay in [8] using methods that are generalized to $\mathcal{N} = 8$ in [7] and in this paper to six supersymmetries. In [9] the $\mathcal{N} = 1$ theory was derived from the superconformal algebra by imposing constraints on some of the field strength components, while in [10] the same methods were used to obtain a superconformal Lagrangian for any $\mathcal{N}$.

In [7] also the problem of coupling the $\mathcal{N} = 8$ conformal supergravity to the $\mathcal{N} = 8$ BLG theory was discussed and the Lagrangian partly derived. Here we will first briefly review the construction of $\mathcal{N} = 8$ pure topological supergravity as presented in [7], and then redo this for $\mathcal{N} = 6$. The goal in the following sections is then to derive the Lagrangian describing the coupling of this $\mathcal{N} = 6$ topological gravity theory to the ABJM theory where we in a first step follow [7].

3.1 $\mathcal{N} = 8$ pure topological supergravity

Following the work of Deser and Kay [8] the authors of [7] constructed the on-shell Lagrangian of three-dimensional $\mathcal{N} = 8$ conformal supergravity using only the three gauge fields of 'spin' 2, 3/2 and 1, i.e., $e_\mu{}^\alpha$, χ_μ, B_μ^{ij}. The result is

$$L = \frac{1}{2}\epsilon^{\mu\nu\rho}Tr_\alpha(\tilde{\omega}_\mu\partial_\nu\tilde{\omega}_\rho + \frac{2}{3}\tilde{\omega}_\mu\tilde{\omega}_\nu\tilde{\omega}_\rho) - \epsilon^{\mu\nu\rho}Tr_i(B_\mu\partial_\nu B_\rho + \frac{2}{3}B_\mu B_\nu B_\rho)$$

$$-ie^{-1}\epsilon^{\alpha\mu\nu}\epsilon^{\beta\rho\sigma}(\tilde{D}_\mu\bar{\chi}_\nu\gamma_\beta\gamma_\alpha\tilde{D}_\rho\chi_\sigma), \tag{3.1}$$

which was in [7] explicitly shown to have $\mathcal{N}=8$ supersymmetry under the following transformation rules of the dreibein and Rarita-Schwinger field:

$$\delta e_\mu{}^\alpha = i\bar{\epsilon}\gamma^\alpha\chi_\mu, \;\; \delta\chi_\mu = \tilde{D}_\mu\epsilon. \tag{3.2}$$

By demanding supersymmetry for any value of the R-symmetry gauge field strength, one immediately concludes that the gauge field must vary under supersymmetry as follows:

$$\delta B^{ij}_\mu = -\frac{i}{2e}\bar{\epsilon}\Gamma^{ij}\gamma_\nu\gamma_\mu f^\nu. \tag{3.3}$$

The covariant derivative appearing in the Lagrangian and in the variation of the Rarita-Schwinger field takes the form

$$\tilde{D}_\mu\epsilon = \partial_\mu\epsilon + \frac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\epsilon + \frac{1}{4}B^{ij}_\mu\Gamma^{ij}\epsilon, \tag{3.4}$$

acting on a three-dimensional spinor in an $SO(8)$ spinor representation.

Thus we explicitly gauge both the $SO(1,2)$ Lorentz and the $SO(8)$ R symmetry. Note that the spinors in the gravity sector, i.e., the SUSY parameter and the Rarita-Schwinger field, are of the same $SO(8)$ chirality while the spinor in the BLG theory is of opposite chirality. The commutator of two supercovariant derivatives, acting on an SO(8) spinor, is

$$[\tilde{D}_\mu, \tilde{D}_\nu] = \frac{1}{4}\tilde{R}_{\mu\nu\alpha\beta}\gamma^{\alpha\beta} + \frac{1}{4}G^{ij}_{\mu\nu}\Gamma^{ij}, \tag{3.5}$$

It will be convenient to define the dual R-symmetry and curvature fields

$$G^{*\mu}_{ij} = \frac{1}{2}\epsilon^{\mu\nu\rho}G_{\nu\rho ij}, \;\; \tilde{R}^{*\mu}{}_{\alpha\beta} = \frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{R}_{\nu\rho\alpha\beta} \tag{3.6}$$

and similarly for $\tilde{\omega}$, as well as the double and triple duals

$$\tilde{R}^{**\mu,\alpha} = \frac{1}{2}\epsilon^{\alpha\beta\gamma}\tilde{R}^{*\mu}{}_{\beta\gamma}, \;\; \tilde{R}^{***}_\mu = \frac{1}{2}\epsilon_{\mu\nu\alpha}\tilde{R}^{**\nu,\alpha}. \tag{3.7}$$

Also as in [8], we define the spin 3/2 field strength

$$f^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}\tilde{D}_\nu\chi_\rho, \tag{3.8}$$

which can be used to write the spin 3/2 conformal term in the Lagrangian as

$$-4i(e_\mu{}^\alpha e_\nu{}^\beta e^{-1})\bar{f}^\mu\gamma_\beta\gamma_\alpha f^\nu. \tag{3.9}$$

The standard procedure to obtain local supersymmetry is to start by adding Rarita-Schwinger terms to the dreibein-compatible ω in order to obtain a supercovariant version of it. That is, we define

$$\tilde{\omega}_{\mu\alpha\beta} = \omega_{\mu\alpha\beta} + K_{\mu\alpha\beta}, \tag{3.10}$$

where

$$\omega_{\mu\alpha\beta} = \frac{1}{2}(\Omega_{\mu\alpha\beta} - \Omega_{\alpha\beta\mu} + \Omega_{\beta\mu\alpha}), \tag{3.11}$$

with

$$\Omega_{\mu\nu\alpha} = \partial_\mu e_\nu{}^\alpha - \partial_\nu e_\mu^\alpha, \tag{3.12}$$

and

$$K_{\mu\alpha\beta} = -\frac{i}{2}(\bar{\chi}_\mu\gamma_\beta\chi_\alpha - \bar{\chi}_\mu\gamma_\alpha\chi_\beta - \bar{\chi}_\alpha\gamma_\mu\chi_\beta). \tag{3.13}$$

This combination of spin connection and torsion is supercovariant, i.e., derivatives on the supersymmetry parameter cancel out if $\tilde{\omega}_{\mu\alpha\beta}$ is varied under the ordinary transformations of the dreibein and Rarita-Schwinger field.

In [7] the supersymmetry of the Lagrangian given above for $\mathcal{N} = 8$ conformal supergravity was demonstrated in full detail which required a certain amount of Fierz transformations. We will not discuss this further here. Instead we turn to the $\mathcal{N} = 6$ case and give some of the details in that context.

3.2 $\mathcal{N} = 6$ pure topological supergravity

Let us start from the fact that in the ABJM theory [5] the supersymmetry parameter is written ϵ_{AB}, with two antisymmetric $SU(4)$ indices in the fundamental representation, thus producing six complex components. To get a parameter in the real six-dimensional vector representation of $SU(4) = SO(6)$ we need to impose the complex self-duality condition (recall that $\epsilon^{AB} = (\epsilon_{AB})^*$)[3]

$$\epsilon^{AB} = \tfrac{1}{2}\epsilon^{ABCD}\epsilon_{CD}. \tag{3.14}$$

With these conventions the local supersymmetry transformations take the form

$$\delta e_\mu{}^\alpha = i\bar{\epsilon}^{AB}\gamma^\alpha\chi_{\mu AB}, \;\; \delta\chi_{\mu AB} = \tilde{D}_\mu\epsilon_{AB}. \tag{3.15}$$

Our goal now is to find a conformal Lagrangian that is supersymmetric under the above $\mathcal{N} = 6$ transformations of the dreibein and the Rarita-Schwinger field together with a transformation of the $SO(6)$ R-symmetry gauge field $B_\mu{}^A{}_B$ that will be determined in the course of the calculation. This superconformal $\mathcal{N} = 6$ supergravity theory will then be coupled to the ABJM theory in later sections.

As we will show below the Lagrangian is the same as for $\mathcal{N} = 8$ apart from the normalization of the R-symmetry Chern-Simons term which differs by a factor of two. This is due to the fact that the trace is over the fundamental $SU(4)$ representation (indices $A, B, ...$) instead of the vector representation as in the $\mathcal{N} = 8$ case. Thus we claim that the Lagrangian for $\mathcal{N} = 6$ is

$$L = \frac{1}{2}\epsilon^{\mu\nu\rho}Tr_\alpha(\tilde{\omega}_\mu\partial_\nu\tilde{\omega}_\rho + \frac{2}{3}\tilde{\omega}_\mu\tilde{\omega}_\nu\tilde{\omega}_\rho) - 2\epsilon^{\mu\nu\rho}Tr_A(B_\mu\partial_\nu B_\rho + \frac{2}{3}B_\mu B_\nu B_\rho)$$

[3]See the previous footnote.

$$-ie^{-1}\epsilon^{\alpha\mu\nu}\epsilon^{\beta\rho\sigma}(\tilde{D}_\mu\bar{\chi}_\nu^{AB}\gamma_\beta\gamma_\alpha\tilde{D}_\rho\chi_{\sigma AB}), \tag{3.16}$$

where the last term can also be written

$$-4i(e_\mu{}^\alpha e_\nu{}^\beta e^{-1})\bar{f}^{\mu AB}\gamma_\beta\gamma_\alpha f^\nu_{AB}, \tag{3.17}$$

in terms of the Rarita-Schwinger field strength f^μ_{AB} defined as in the $\mathcal{N}=8$ case discussed above.

The covariant derivative acting on for instance the susy parameter is defined by

$$\tilde{D}_\mu\epsilon_{AB} = \partial_\mu\epsilon_{AB} + \tfrac{1}{4}\tilde{\omega}_{\mu\alpha\beta}\gamma^{\alpha\beta}\epsilon_{AB} - B_\mu{}^C{}_A\epsilon_{CB} - B_\mu{}^C{}_B\epsilon_{AC}. \tag{3.18}$$

By demanding that terms proportional to the R-symmetry gauge field strength cancel among themselves we find the following transformation rule for the B_μ field

$$\delta B_\mu{}^A{}_B = \tfrac{i}{e}(\bar{f}^{AC}_\sigma\gamma_\mu\gamma^\sigma\epsilon_{BC} - \bar{f}^\sigma_{BC}\gamma_\mu\gamma_\sigma\epsilon^{AC}). \tag{3.19}$$

This expression can also be written

$$\delta B_\mu{}^A{}_B = \tfrac{2i}{e}(\bar{f}^{AC}_\sigma\gamma_\mu\gamma^\sigma\epsilon_{BC} - \tfrac{1}{4}\delta^A_B\bar{f}^{CD}_\nu\gamma_\mu\gamma^\nu\epsilon^{CD}), \tag{3.20}$$

and hence is defined to be traceless (see comment at the end of this subsection).

The calculation now goes through exactly as for $\mathcal{N}=8$, using for instance expressions like

$$\delta\tilde{\omega}^*_{\mu,\nu} = -2i(\bar{\epsilon}^{AB}\gamma_\mu f_{\nu AB} - \tfrac{1}{2}g_{\mu\nu}\bar{\epsilon}^{AB}\gamma^\rho f_{\rho AB}) \tag{3.21}$$

and leads to the following expression for δL:

$$\begin{aligned}\delta L = & \tfrac{4}{e}\bar{\epsilon}^{AB}(\gamma_\mu\gamma_\nu f^\mu_{AB})\bar{f}^{CD}_\sigma\gamma^\nu\chi^\sigma_{CD}\\ & +\tfrac{8}{e}\bar{f}^\mu_{CD}(\gamma_\nu\gamma_\alpha f^{\nu CD})(\bar{\epsilon}_{AB}\gamma^\alpha\chi^{AB}_\mu - \tfrac{1}{2}e_\mu{}^\alpha\bar{\epsilon}_{AB}\gamma^\sigma\chi^{AB}_\sigma),\\ & +\tfrac{4}{e^2}(\bar{f}^\mu_{AB}\gamma_\nu\gamma_\mu)\gamma_\gamma\chi^{AB}_\rho\epsilon^{\nu\rho\sigma}(\bar{\epsilon}^{CD}\gamma_\sigma f^\gamma_{CD} - \tfrac{1}{2}e_\sigma{}^\gamma\bar{\epsilon}^{CD}\gamma_\tau f^\tau_{CD}),\\ & -\tfrac{16}{e^2}(\bar{f}^{\mu AB}\gamma_\nu\gamma_\mu)\chi_{\sigma CB}\epsilon^{\nu\rho\sigma}\bar{\epsilon}_{AD}(\gamma_\tau\gamma_\rho f^{\tau CD})\\ & +\tfrac{8}{e^2}(\bar{f}^{\mu AB}\gamma_\nu\gamma_\mu)\chi_{\sigma AB}\epsilon^{\nu\rho\sigma}\bar{\epsilon}_{CD}(\gamma_\tau\gamma_\rho f^{\tau CD}).\end{aligned} \tag{3.22}$$

As in the $\mathcal{N}=8$ case presented in detail in [7], to demonstrate supersymmetry we need to Fierz this expression and show that it vanishes. However, at this point we will diverge from the treatment of the $\mathcal{N}=8$ theory where both the $SO(1,2)$ and the $SO(8)$ spinor indices were Fierzed together. Here we first Fierz only the spacetime $SO(1,2)$ spinors and then instead apply representation theory arguments or alternatively cycling of the $\mathcal{N}=6$ spinor indices to conclude the proof of supersymmetry.

The strategy is thus to use the three-dimensional Fierz identity

$$\bar{A}B\bar{C}D = -\tfrac{1}{2}(\bar{A}D\bar{C}B + \bar{A}\gamma^\mu D\bar{C}\gamma_\mu B) \tag{3.23}$$

to write all terms in δL above in a form similar to the second term, i.e., with the two f^μ_{AB} in the same scalar factor. The result of this operation is a number of terms similar to the second term but with the $SU(4)$ indices appearing in various positions: The two f^μ_{AB} can have both indices contracted (as in the second term) as well as one (from the fourth term) or no (from the remaining terms) contracted indices.

To understand how these different terms are related to each other it is convenient to recall from the appendix of ref.[7] that the terms in δL can be Fierzed into a combination of twelve mutually independent expressions (disregarding for the moment the $SU(4)$ indices) of the type $(\bar{f}_\mu f_\nu)(\bar{\epsilon}\chi_\rho)\epsilon^{\mu\nu\rho}$, $(\bar{f}_\mu \gamma^\nu f^\mu)(\bar{\epsilon}\chi_\nu)$ etc. Then considering the fact that $\mathbf{6}\times\mathbf{6}=\mathbf{1}+\mathbf{15}+\mathbf{20}$ where, if written in terms of four fundamental indices, $\mathbf{15}$ is antisymmetric and $\mathbf{1}$ and $\mathbf{20}$ are symmetric under an interchange of the two antisymmetric pairs of indices. Using these properties all terms in δL with the expression $\bar{f}...f$ in any given representation can be collected and seen to cancel exactly.

A second way to obtain this result arises if we consider the fact that the anti-symmetrization of five SU(4) indices vanishes. We can then relate all terms with different index structures to the three independent ones $\mathbf{1}$, $\mathbf{15}$ and $\mathbf{20}$, which can then be collected and seen to cancel separately.

This theory will only be supersymmetric if the gauged R-symmetry is $SU(4)$, i.e., trying to include an abelian factor does not work. This can be seen for instance by making use of the equation

$$\bar{f}^{AC}\chi_{BC} = -\bar{f}_{BC}\chi^{AC} + \tfrac{1}{2}\delta^A_B \bar{f}^{CD}\chi_{CD}, \tag{3.24}$$

that is a direct consequence of the self-duality properties of the two fields in the equation. Note that this particular combination of f and χ appears for instance in the Chern-Simons term for the gravitino field where it is contracted with an R-symmetry gauge field. Demanding that this term in the Lagrangian is real implies, due to the second term on the right hand side above, that the $B_\mu{}^A{}_B$ field is traceless. The term that removes the trace is then responsible for the very last term in expression for δL presented above, and is needed in order to conclude that all terms cancel.

Similarly to the SO(8) case, the theory considered here also has local scale invariance (denoted by an index Δ) and possesses $\mathcal{N}=6$ superconformal (shift) symmetry (denoted by S) with the following transformation rules (where ϕ is the local infinitesimal scale parameter and η the local shift parameter)

$$\begin{aligned}
\delta_\Delta e_\mu{}^\alpha &= -\phi(x)e_\mu{}^\alpha,\\
\delta_\Delta \chi^{AB}_\mu &= -\tfrac{1}{2}\phi(x)\chi^{AB}_\mu,\\
\delta_\Delta B^A_\mu{}_B &= 0,
\end{aligned} \tag{3.25}$$

and

$$\delta_S e_\mu{}^\alpha = 0,$$

$$\begin{aligned} \delta_S \chi_\mu^{AB} &= \gamma_\mu \eta^{AB}, \\ \delta_S B_\mu{}^A{}_C &= -i(\bar{\eta}^{AB}\chi_{\mu BC} - \bar{\chi}_\mu^{AB}\eta_{BC}). \end{aligned} \tag{3.26}$$

4. The $\mathcal{N} = 6$ ungauged ABJM theory

In this section we review the (ungauged) superconformal matter sector, i.e., the ordinary ABJM theory, to which we would like to couple the superconformal gravity derived in the previous section. The resulting "topologically gauged" ABJM theory is then the subject of the following sections.

4.1 Review of the ungauged $\mathcal{N} = 6$ superconformal ABJM action

The formulation of the $\mathcal{N} = 6$ matter theory in [5] makes no reference at all to any three-algebra structure constants in contrast to the situation for the $\mathcal{N} = 8$ BLG theory. However, as shown in [11] the ABJM theory is easily rewritten in terms of such structure constants, a fact that was further developed in [12] where the theory was expressed in terms of an additional algebraic structure related to generalized Jordan triple systems. This provides a new interpretation of the index structure of the fields and the structure constants in terms of an infinitely graded Lie algebra[4]. The particular form of the ABJM action that we find convenient to use here is presented in [12].

In this new form of ABJM action, the complex scalars and fermions are defined to have the specific index structure Z_a^A and Ψ_{Aa}, while their complex conjugates have the index structure Z_A^a and Ψ^{Aa}. These fields are then connected to a formulation of the theory where the structure constants have two upper and two lower indices [12]. Furthermore, these indices are antisymmetric in each pair separately

$$f^{ab}{}_{cd} = f^{[ab]}{}_{cd} = f^{ab}{}_{[cd]}\,. \tag{4.1}$$

The action of the $\mathcal{N}$= 6 M2-theory can now be written as follows:

$$\begin{aligned} \mathcal{L} = &-(D_\mu Z_a^A)(D^\mu \bar{Z}_A{}^a) - i\bar{\Psi}^{Aa}\gamma^\mu D_\mu \Psi_{Aa} \\ &- i f^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Aa} Z_b^B \bar{Z}_B^c + 2i f^{ab}{}_{cd}\bar{\Psi}^{Ad}\Psi_{Ba} Z_b^B \bar{Z}_A^c \\ &- \tfrac{i}{2}\epsilon_{ABCD} f^{ab}{}_{cd}\bar{\Psi}^{Ac}\Psi^{Bd} Z_a^C Z_b^D - \tfrac{i}{2}\epsilon^{ABCD} f^{cd}{}_{ab}\bar{\Psi}_{Ac}\Psi_{Bd}\bar{Z}_C^a \bar{Z}_D^b \\ &- V + \tfrac{1}{2}\epsilon^{\mu\nu\lambda}(f^{ab}{}_{cd}A_\mu{}^d{}_b \partial_\nu A_\lambda{}^c{}_a + \tfrac{2}{3} f^{bd}{}_{gc} f^{gf}{}_{ae} A_\mu{}^a{}_b A_\nu{}^c{}_d A_\lambda{}^e{}_f)\,, \end{aligned} \tag{4.2}$$

where the gauge fields $A_\mu{}^a{}_b$ naturally appear in the covariant derivatives in the following form

$$\tilde{A}_\mu{}^a{}_b = f^{ac}{}_{bd} A_\mu{}^d{}_c\,, \tag{4.3}$$

and the potential takes the form

$$V = \tfrac{2}{3}\Upsilon^{CD}{}_{Bd}\bar{\Upsilon}_{CD}{}^{Bd}\,, \tag{4.4}$$

[4]This algebra is further discussed in [13].

$$\Upsilon^{CD}{}_{Bd} = f^{ab}{}_{cd} Z^C_a Z^D{}_b \bar{Z}^c_B + f^{ab}{}_{cd} \delta^{[C}{}_B Z^{D]}_a Z^E_b \bar{Z}^c_E \,. \tag{4.5}$$

The transformation rules for the six supersymmetries, parametrized by the complex self-dual three-dimensional spinor ϵ_{AB}, read:

$$\delta Z^A_a = i\bar{\epsilon}^{AB}\Psi_{Ba}\,, \tag{4.6}$$

$$\delta\Psi_{Bd} = \gamma^\mu D_\mu Z^A_d \epsilon_{AB} + f^{ab}{}_{cd} Z^C_a Z^D_b \bar{Z}^c_B \epsilon_{CD} - f^{ab}{}_{cd} Z^A_a Z^C_b \bar{Z}^c_C \epsilon_{AB}\,, \tag{4.7}$$

$$\delta A_\mu{}^a{}_b = -i\bar{\epsilon}_{AB}\gamma_\mu \Psi^{Aa} Z^B_b + i\bar{\epsilon}^{AB}\gamma_\mu \Psi_{Ab}\bar{Z}^a_B\,. \tag{4.8}$$

This action can be shown to be $\mathcal{N}=6$ supersymmetric provided that the structure constants obey the fundamental identity [12] (see also [11])

$$f^{a[b}{}_{dc} f^{e]d}{}_{gh} = f^{be}{}_{d[g} f^{ad}{}_{h]c}\,, \tag{4.9}$$

and, under complex conjugation,

$$(f^{ab}{}_{cd})^* = f^{cd}{}_{ab} \equiv f_{ab}{}^{cd}. \tag{4.10}$$

5. Coupling $\mathcal{N}=6$ conformal supergravity to ABJM matter: the result after cancelation of $(D_\mu)^2$ terms in δL

In the two previous sections we have discussed both the ABJM theory and $\mathcal{N}=6$ conformal supergravity, the latter derived explicitly in section three. The coupling of these two theories to each other can be obtained in several ways. Here we will use the method based on an expansion in powers of derivatives used previously in ref. [7]. Thus, as the first step we consider in this section only the cancelation of terms in the variation of the Lagrangian that are of second order in covariant derivatives. Terms of third order in derivatives also appear but only in the supergravity sector and have thus already been analyzed. This procedure was demonstrated in [7] to produce additional terms in the transformation rules for the spin one gauge fields in addition to a set of coupling terms that render the theory supersymmetric to this order in covariant derivatives. Applying this strategy here we use the following terms as a starting point:

$$L = L^{conf.}_{sugra} + L^{cov.}_{ABJM} + L^{cov}_{supercurrent}, \tag{5.1}$$

where $L^{conf.}_{sugra}$ has been given in a previous section, the covariantized ABJM Lagrangian

$$L^{cov.}_{ABJM} = e(-(\tilde{D}_\mu Z^A_a)(\tilde{D}^\mu \bar{Z}^a_A) - i\bar{\Psi}^{Aa}\gamma^\mu \tilde{D}_\mu \Psi_{Aa} + L_{Yukawa} - V) + L_{CS(A)}, \tag{5.2}$$

and

$$L^{cov}_{supercurrent} = Aie(\bar{\chi}_{\mu AB}\gamma^\nu\gamma^\mu\Psi^{Ba})(\tilde{D}_\nu Z^A_a - \tfrac{i}{2}\hat{A}\bar{\chi}^{AD}_\nu \Psi_{Da}) + c.c., \tag{5.3}$$

where the constants A and $\hat{A}$ will be determined below.

The transformation rules at this point in the analysis are the ones used in sections three and four but with fully covariant derivatives, reproduced here for convenience,

$$\begin{aligned}
\delta e_\mu{}^\alpha &= i\bar{\epsilon}_g^{AB}\gamma^\alpha\chi_{\mu AB}\,,\\
\delta\chi^{\mu AB} &= \tilde{D}_\mu\epsilon_g^{AB}\,,\\
\delta B_\mu{}^A{}_B &= \tfrac{i}{e}(\bar{f}_\sigma^{AC}\gamma_\mu\gamma^\sigma\epsilon_{gBC} - \bar{f}^\sigma_{BC}\gamma_\mu\gamma_\sigma\epsilon_g^{AC})\,,\\
\delta Z_a^A &= i\bar{\epsilon}_m^{AB}\Psi_{Ba}\,,\\
\delta\Psi_{Bd} &= \gamma^\mu(\tilde{D}_\mu Z_d^A - i\hat{A}\bar{\chi}_\mu^{AD}\Psi_{Dd})\epsilon_{mAB} + f^{ab}{}_{cd}Z_a^C Z_b^D \bar{Z}_B^c\epsilon_{mCD} - f^{ab}{}_{cd}Z_a^A Z_b^C \bar{Z}_C^c\epsilon_{mAB}\,,\\
\delta A_\mu{}^a{}_b &= -i\bar{\epsilon}_{mAB}\gamma_\mu\Psi^{Aa}Z_b^B + i\bar{\epsilon}_m^{AB}\gamma_\mu\Psi_{Ab}\bar{Z}_B^a\,,
\end{aligned}\tag{5.4}$$

where the two (gravity and matter) supersymmetry parameters will be related below.

We will later need to add more terms in order to keep the theory supersymmetric to the order of approximation we are then working. Note, however, that the hatted coefficient $\hat{A}$ in the ansatz is not determined by the $(\tilde{D}_\mu)^2$ calculation below but simply by demanding that the $\tilde{D}_\mu Z$ factor in $\delta\Psi$ be supercovariant, i.e., $\tilde{D}_\mu Z$ must be replaced, as done in the ansatz, by $\tilde{D}_\mu Z - i\hat{A}\bar{\chi}\Psi$ in order to eliminate terms where the derivative acts on the supersymmetry parameter when this expression is varied. The parameter $\hat{A}$ is then obtained as soon as the relation between the ABJM and supergravity supersymmetry parameters are determined. Note that the presence of a factor of $\frac{1}{2}$ in front of $\hat{A}$ in the supercurrent term is common in supergravity and follows from standard arguments. These features of the theory will be verified in the next chapter when supersymmetry is implemented by canceling terms in δL with one derivative.

5.1 Supersymmetry at order $(\tilde{D}_\mu)^2$

We start by performing the variation of the covariantized scalar and spinor kinetic terms. The scalar one

$$L_1 = -eg^{\mu\nu}(\tilde{D}_\mu Z_a^A)(\tilde{D}_\nu \bar{Z}_A^a),\tag{5.5}$$

gives

$$\begin{aligned}
\delta L_1 = &\,2ie(\tilde{D}_\mu Z_a^A)(\tilde{D}_\nu\bar{Z}_A^a)(\bar{\epsilon}_g^{BC}\gamma^{(\mu}\chi^{\nu)}_{BC} - \tfrac{1}{2}g^{\mu\nu}\bar{\epsilon}_g^{BC}\gamma^\rho\chi_{\rho BC})\\
&+ie(\bar{\epsilon}_m^{AB}\Psi_{Ba}\tilde{\Box}\bar{Z}_A^a + \tilde{\Box}Z_a^A\bar{\epsilon}_{mAB}\Psi^{Ba})\\
&-eg^{\mu\nu}(-Z_b^A\delta\tilde{A}_\mu{}^b{}_a + \delta B_\mu{}^A{}_B Z_a^B)\tilde{D}_\nu\bar{Z}_A^a\\
&-eg^{\mu\nu}\tilde{D}_\mu Z_a^A(\delta\tilde{A}_\nu{}^a{}_b\bar{Z}_A^b - \bar{Z}_B^a\delta B_\nu{}^B{}_A).
\end{aligned}\tag{5.6}$$

Our first goal will be to cancel the first two lines. For the second line we need the variation of the Dirac term

$$L_2 = -iee_\alpha{}^\mu\bar{\Psi}^{Aa}\gamma^\alpha\tilde{D}_\mu\Psi_{Aa}.\tag{5.7}$$

Its variation reads, after an integration by parts which produces a torsion term $\tilde{D}_\mu e_\alpha{}^\mu = K_{\mu\alpha}{}^\mu = \bar{\chi}_\alpha^{BC}\gamma^\beta\chi_{\beta BC}$ (second line),

$$\delta L_2 = 2e\bar{\epsilon}_g^{BC}\gamma^\beta\chi_{\rho BC}e_{[\alpha}{}^\mu e_{\beta]}{}^\rho\bar{\Psi}^{Aa}\gamma^\alpha\tilde{D}_\mu\Psi_{Aa}$$

$$
\begin{aligned}
&+e(\bar\chi_\alpha^{BC}\gamma^\beta\chi_{\beta BC})\bar\Psi_{Aa}\gamma^\alpha\delta\Psi^{Aa}\\
&-ie(\bar\Psi^{Aa}\gamma^\mu\tilde D_\mu\delta\Psi_{Aa}+\bar\Psi_{Aa}\gamma^\mu\tilde D_\mu\delta\Psi^{Aa})\\
&-ie\bar\Psi^{Aa}\gamma^\mu(\tfrac14\delta\tilde\omega_{\mu\alpha\beta}\gamma^{\alpha\beta}\Psi_{Aa}+\delta\tilde A_{\mu a}{}^b\Psi_{Ab}+\delta B_{\mu A}{}^B\Psi_{Ba}).
\end{aligned}
\tag{5.8}
$$

Thus we need to compute $\gamma^\nu\tilde D_\nu\delta\Psi_{Bd}$. We find, again using $\tilde D_\nu e_\alpha{}^\mu=K_{\nu\alpha}{}^\mu$,

$$
\begin{aligned}
\gamma^\nu\tilde D_\nu\delta\Psi_{Bd} = {}&(\gamma^\nu\gamma^\mu\tilde D_\nu\tilde D_\mu Z_d^A)\epsilon_{mAB}+\gamma^\nu\gamma^\mu(\tilde D_\nu\epsilon_{mAB})(\tilde D_\mu Z_d^A)+\gamma^\nu\gamma^\alpha\epsilon_{mAB}(\tilde D_\mu Z_d^A)K_{\nu\alpha}{}^\mu\\
&-i\hat A\gamma^\nu\gamma^\alpha\epsilon_{mAB}K_{\nu\alpha}{}^\mu(\bar\chi_\mu^{AD}\Psi_{Dd})-i\hat A\gamma^\nu\gamma^\mu\tilde D_\nu\epsilon_{mAB}(\bar\chi_\mu^{AD}\Psi_{Dd})\\
&-i\hat A\gamma^\nu\gamma^\mu\epsilon_{mAB}(\tilde D_\nu\bar\chi_\mu^{AD}\Psi_{Dd}+\bar\chi_\mu^{AD}\tilde D_\nu\Psi_{Dd})\\
&+\gamma^\nu\epsilon_{mCD}f^{ab}{}_{cd}(2(\tilde D_\nu Z_a^C)Z_b^D\bar Z_B^c+Z_a^CZ_b^D(\tilde D_\nu\bar Z_B^c))\\
&-\gamma^\nu\epsilon_{mAB}f^{ab}{}_{cd}((\tilde D_\nu Z_a^A)Z_b^C\bar Z_C^c+Z_a^A(\tilde D_\nu Z_b^C)\bar Z_C^c+Z_a^AZ_b^C(\tilde D_\nu\bar Z_C^c))\\
&+f^{ab}{}_{cd}(\gamma^\nu\tilde D_\nu\epsilon_{mCD}Z_a^CZ_b^D\bar Z_B^c-\gamma^\nu\tilde D_\nu\epsilon_{mAB}Z_a^AZ_b^C\bar Z_C^c)\,.
\end{aligned}
\tag{5.9}
$$

Now since

$$
\gamma^\nu\gamma^\mu\tilde D_\nu\tilde D_\mu Z_d^A=\tilde\Box Z_d^A+\tfrac12\gamma^{\mu\nu}\tilde{(}F_{\mu\nu d}{}^eZ_e^A+G_{\mu\nu}{}^A{}_BZ_d^B)
\tag{5.10}
$$

we see that the box terms from the variations of the scalar and spinor kinetic terms cancel.

Next we concentrate on the first line of the variation of the scalar kinetic term above and the second term in the variation of the Dirac operator. To cancel these two terms one needs to introduce the supercurrent term. Our ansatz for this term reads

$$
\begin{aligned}
L_{SC_{1,2}} = {}&-Ai(ee_\alpha{}^\mu e_\beta{}^\nu)\bar\chi_\mu^{AB}\gamma^\beta\gamma^\alpha\Psi_{Aa}(\tilde D_\nu\bar Z_B^a-\tfrac{i}{2}\hat A\bar\chi_{\nu BC}\Psi^{Ca})\\
&-\ Ai(ee_\alpha{}^\mu e_\beta{}^\nu)\bar\chi_{\mu AB}\gamma^\beta\gamma^\alpha\Psi^{Aa}(\tilde D_\nu Z_a^B-\tfrac{i}{2}\hat A\bar\chi_\nu^{BC}\Psi_{Ca})\,,
\end{aligned}
\tag{5.11}
$$

where the index 1 refers to the first term in the two brackets and 2 to the $\hat A$ terms. Terms of the kind we are here seeking to cancel arise if we vary the two spinors in L_{SC_1}. Varying χ gives

$$
-Aie\tilde D_\mu\bar\epsilon_g^{AB}\gamma^\nu\gamma^\mu\Psi_{Aa}\tilde D_\nu\bar Z_B^a+c.c.
\tag{5.12}
$$

while the variation of Ψ produces the result

$$
-Aie\bar\chi_\mu^{AB}\gamma^\nu\gamma^\mu\gamma^\rho\tilde D_\rho Z_a^D\epsilon_{mDA}\tilde D_\nu\bar Z_B^a+c.c.\,.
\tag{5.13}
$$

Using the duality flip and the identity

$$
\gamma^\nu\gamma^\mu\gamma^\rho=\tfrac1e\epsilon^{\nu\mu\rho}+2(g^{\mu(\nu}\gamma^{\rho)}-\tfrac12g^{\nu\rho}\gamma^\mu)
\tag{5.14}
$$

we find that demanding cancelation gives two conditions on the matter and gravity supersymmetry parameters:

$$
2\epsilon_g=A\epsilon_m,\ \ \epsilon_m=A\epsilon_g,\ \ \to A=\pm\sqrt2,\ \ \epsilon_g=\pm\tfrac{1}{\sqrt2}\epsilon_m.
\tag{5.15}
$$

After these cancellations the remaining $\tilde{D}^2$ terms are

$$iA(\bar{\chi}_{\mu}^{AB}\epsilon_{mAD} - \bar{\chi}_{\mu AD}\epsilon_{m}^{AB})\epsilon^{\mu\nu\rho}\tilde{D}_{\nu}Z_a^D\tilde{D}_{\rho}\bar{Z}_B^a\,, \tag{5.16}$$

which forces us to add a χ^2 term to Lagrangian, namely

$$L_{A'} = iA'\epsilon^{\mu\nu\rho}\bar{\chi}_{\mu}^{AC}\chi_{\nu BC}Z_A^a\tilde{D}_{\rho}\bar{Z}_a^B + c.c.\,, \tag{5.17}$$

where $\bar{\chi}\chi$ is automatically in the representation **15** of $SU(4)$ so the derivative can only be integrated by parts onto the other scalar field (reality of this term then follows from the duality flip property). The variation gives, after some integrations by parts,

$$\begin{aligned}
\delta L_{A'} &= -2iA'(\bar{\epsilon}_g^{AC}f_{BC}^{\rho} - \bar{f}^{\rho AC}\epsilon_{gBC})(Z_a^B\tilde{D}_{\rho}\bar{Z}_A^a)\\
&+iA'\epsilon^{\mu\nu\rho}(\bar{\epsilon}_g^{AC}\chi_{\mu BC} - \bar{\chi}_{\mu}^{AC}\epsilon_{gBC})(\tilde{D}_{\nu}Z_a^B\tilde{D}_{\rho}\bar{Z}_A^a + \tfrac{1}{2}Z_a^B\tilde{F}_{\nu\rho}{}^a{}_b\bar{Z}_A^b + \tfrac{1}{2}Z_a^B G_{\nu\rho A}{}^D\bar{Z}_D^a)\\
&-A'\epsilon^{\mu\nu\rho}\bar{\chi}_{\mu}^{AC}\chi_{\nu BC}(\bar{\epsilon}_m^{BD}\Psi_{Da}(\tilde{D}_{\rho}\bar{Z}_A^a) + Z_a^B(\tilde{D}_{\rho}\bar{\epsilon}_{mAD})\Psi^{Da} + Z_a^B\bar{\epsilon}_{mAD}(\tilde{D}_{\rho}\Psi^{Da})).\\
&+c.c.\,.
\end{aligned} \tag{5.18}$$

Thus we find that, provided $A' = 1$, the term $\tilde{D}_{\nu}Z_a^B\tilde{D}_{\rho}\bar{Z}_A^a$ and its complex conjugate cancel the same terms previously obtained in the variation of the supercurrent.

This leaves us with the following $\tilde{D}^2$ terms

$$-2iA'(\bar{\epsilon}_g^{AC}f_{BC}^{\rho} - \bar{\epsilon}_{gBC}f^{\rho AC})Z_a^B\tilde{D}_{\rho}\bar{Z}_A^a + c.c.\,. \tag{5.19}$$

If we now consider the terms obtained by performing the gravitational variation of the R-symmetry gauge field in the Klein-Gordon term, we find

$$\delta L_1|_{\delta B_{\mu}|_{grav}} = -i(\bar{f}^{\nu AC}\gamma^{\mu}\gamma_{\nu}\epsilon_{gBC} - \bar{f}_{BC}^{\nu}\gamma^{\mu}\gamma_{\nu}\epsilon_g^{AC})(Z_a^B(\tilde{D}_{\mu}\bar{Z}_A^a) - (\tilde{D}_{\mu}Z_a^B)\bar{Z}_A^a), \tag{5.20}$$

which is entirely a contribution to the $\tilde{D}^2$ terms. Using the fact that $A' = 1$ found above, this expression can be combined with the one in the previous paragraph leaving the following $\tilde{D}^2$ terms in δL

$$i(\bar{f}^{\nu AC}\gamma_{\nu}\gamma^{\mu}\epsilon_{gBC} - \bar{f}_{BC}^{\nu}\gamma_{\nu}\gamma^{\mu}\epsilon_g^{AC})(Z_a^B(\tilde{D}_{\mu}\bar{Z}_A^a) - (\tilde{D}_{\mu}Z_a^B)\bar{Z}_A^a). \tag{5.21}$$

The next term to be added is

$$L_{A''} = iA''(\bar{f}^{AB}\cdot\gamma\Psi_{Aa}\bar{Z}_B^a + \bar{f}_{AB}\cdot\gamma\Psi^{Aa}Z_a^B). \tag{5.22}$$

If we concentrate on the $\tilde{D}^2$ terms we get two such from the variation of χ in f and of Ψ. The former gives

$$\begin{aligned}
&\tfrac{i}{2}A''\epsilon^{\mu\nu\rho}\tilde{D}_{\nu}\tilde{D}_{\rho}\bar{\epsilon}_g^{AB}\gamma_{\mu}\Psi_{Aa}\bar{Z}_B^a + c.c.\\
&= \tfrac{i}{4}A''\epsilon^{\mu\nu\rho}(-\tfrac{1}{4}\tilde{R}_{\nu\rho\alpha\beta}\bar{\epsilon}_g^{AB}\gamma^{\alpha\beta} + 2G_{\nu\rho}{}^{[A}{}_C\bar{\epsilon}_g^{|C|B]})\gamma_{\mu}\Psi_{Aa}\bar{Z}_B^a + c.c.\,,
\end{aligned} \tag{5.23}$$

while the latter generates the expression

$$\begin{aligned}&iA''\bar{f}^{BA}_\nu\cdot\gamma^\nu\gamma^\mu(\tilde{D}_\mu Z^C_d\epsilon_{mCB}-i\hat{A}\bar{\chi}^{CD}_\mu\Psi_{Dd}\epsilon_{mCB}\\&+f^{ab}{}_{cd}Z^C_aZ^D_b\bar{Z}^c_B\epsilon_{mCD}-f^{ab}{}_{cd}Z^D_aZ^C_b\bar{Z}^c_C\epsilon_{mDB})\bar{Z}^d_A+c.c,\end{aligned}\tag{5.24}$$

where the first term is a $\tilde{D}^2$ term, which for $A''=\mp\sqrt{2}$ exactly cancels the traceless part of the previous expression above leaving just the trace part:

$$\begin{aligned}&-\tfrac{iA''}{4}(\bar{f}^{AB}\cdot\gamma\gamma^\mu\epsilon_{mAB})(\tilde{D}_\mu(Z\bar{Z}))\\&=-\tfrac{iA''}{4}(\bar{\epsilon}_{gAB}f^{\mu AB}-\epsilon^{\mu\sigma\rho}\bar{\epsilon}_{gAB}\gamma_\sigma f^{AB}_\rho)(\tilde{D}_\mu(Z\bar{Z})).\end{aligned}\tag{5.25}$$

The next term we need is

$$L_{RZ^2}=-\tfrac{e}{8}\tilde{R}|Z|^2\tag{5.26}$$

which has the following variation

$$\begin{aligned}\delta\tilde{L}_{RZ^2}=&\tfrac{i}{4e}|Z|^2\tilde{R}^{**}_{\mu,\nu}\bar{\epsilon}^{AB}_g\gamma^\nu\chi^\mu_{AB}-\tfrac{i}{4e}\tilde{R}^{**}(\bar{\epsilon}^{AB}_m\Psi_{Aa}\bar{Z}^a_B+\bar{\epsilon}_{mAB}\Psi^{Aa}Z^B_a)\\&+\tfrac{i}{2e}\epsilon^{\mu\nu\rho}(\tilde{D}_\mu|Z|^2+K_{\sigma\mu}{}^\sigma|Z|^2)\bar{\epsilon}^{AB}_g\gamma_\nu f_{\rho AB}\\&+\tfrac{i}{2e}\epsilon^{\mu\nu\rho}K_{\mu\nu}{}^\sigma|Z|^2(\bar{\epsilon}^{AB}_g\gamma_\sigma f_{\rho AB}-\tfrac{1}{2}g_{\sigma\rho}\bar{\epsilon}^{AB}_g\gamma\cdot f_{AB}),\end{aligned}\tag{5.27}$$

where we used the fact that

$$\delta\tilde{\omega}^{*\alpha\beta}_\nu=\tfrac{2i}{e^2}\epsilon^{\alpha\beta\rho}(\bar{\epsilon}^{AB}_g\gamma_\nu f_{\rho AB}-\tfrac{1}{2}g_{\nu\rho}\bar{\epsilon}^{AB}_g\gamma_\sigma f^\sigma_{AB}),\tag{5.28}$$

and that the double dual of the Ricci tensor with torsion

$$\tilde{R}^{**}_{\nu,\mu}=\tilde{R}_{\mu,\nu}-\tfrac{1}{2}g_{\mu\nu}\tilde{R},\tag{5.29}$$

where one should note the order of the indices.

Thus we find that two new cancellations occur between the second and the third terms in this expression and the corresponding ones proportional to A'' above. This leaves three terms at the $\tilde{D}^2$ level to discuss. One is proportional to the R-symmetry gauge field and generates an additional contribution to the variation of the gauge field. The other two are

$$\tfrac{i}{4e}|Z|^2\tilde{R}^{**}_{\mu,\nu}\bar{\epsilon}^{AB}_g\gamma^\nu\chi^\mu_{AB}-\tfrac{i}{4}A''(\bar{\epsilon}^{AB}_g f^\mu_{AB})(\tilde{D}_\mu(Z\bar{Z})).\tag{5.30}$$

The final term we need to add in order to demonstrate that all $\tilde{D}^2$ terms cancel in δL is the fermionic analog of $L_{R|Z|^2}$, namely

$$L_{Z^2f\chi}=iA'''|Z|^2\bar{f}^\mu_{AB}\chi^{AB}_\mu\tag{5.31}$$

whose variation reads

$$\delta L_{Z^2f\chi}=A'''(\bar{\epsilon}_{AB}\Psi^{Aa}Z^B_a+\bar{\epsilon}^{AB}\Psi_{Aa}\bar{Z}^a_B)\bar{f}^\mu\chi^{AB}_\mu-iA'''(\tilde{D}_\mu|Z|^2)\bar{f}^\mu_{AB}\epsilon^{AB}_g$$

$$
\begin{aligned}
&-\tfrac{i}{2}A'''|Z|^2(\tilde{R}^{**\mu,\nu}\bar{\epsilon}_{gAB}\gamma_\nu\chi_\mu^{AB} - 2\epsilon^{\mu\nu\rho}G_{\mu\nu}{}^A{}_C\bar{\epsilon}_g^{CB}\chi_{\rho AB})\\
&+\tfrac{i}{2}A'''|Z|^2\epsilon^{\mu\nu\rho}(-\tfrac{1}{4}\delta\tilde{\omega}_{\nu\alpha\beta}\bar{\chi}_{\rho AB}\gamma^{\alpha\beta}\chi_\mu^{AB} + 2\delta B_\nu{}^A{}_C\bar{\chi}_\rho^{CB}\chi_{\mu AB}).
\end{aligned} \tag{5.32}
$$

Thus we see that the last $\tilde{D}^2$ terms in the previous paragraph are canceled by choosing $A''' = \frac{1}{2}$ and the theory is hence supersymmetric at this order and above in covariant derivarives. Note that in the two-star curvature term only the symmetric part is $\tilde{D}^2$.

To cancel the $G_{\mu\nu}$-terms and $F_{\mu\nu}$-terms we obtained above, it is necessary to add new terms to the variations of the gauge fields A_μ and B_μ, chosen such that their Chern-Simons terms give exactly the same $G_{\mu\nu}$-terms and $F_{\mu\nu}$-terms but with the opposite sign.

Before analysing the terms with less than two derivatives we summarize what we have found so far:

$$
\begin{aligned}
L|_{\tilde{D}^3,\tilde{D}^2} = &\; L^{conf}_{sugra} + L^{cov}_{BLG} + iA(ee_\alpha{}^\mu e_\beta{}^\nu)\bar{\chi}_\mu^{AB}\gamma^\beta\gamma^\alpha\Psi_{Aa}(\tilde{D}_\nu\bar{Z}^a_B - \tfrac{i}{2}\hat{A}\bar{\chi}_{\nu BC}\Psi^{Ca}) + c.c.\\
&+iA'\epsilon^{\mu\nu\rho}\bar{\chi}_\mu^{AC}\chi_{\nu BC}Z^a_A\tilde{D}_\rho\bar{Z}_a^B + c.c.\\
&+iA''(\bar{f}^{\mu AB}\gamma_\mu\Psi_{Aa}\bar{Z}^a_B + \bar{f}^\mu_{AB}\gamma_\mu\Psi^{Aa}Z_a^B)\\
&-\tfrac{e}{8}\tilde{R}|Z|^2 + iA'''|Z|^2\bar{f}^\mu_{AB}\chi_\mu^{AB}
\end{aligned} \tag{5.33}
$$

is supersymmetric to second and third order in derivatives under the following transformation rules

$$
\begin{aligned}
\delta e_\mu{}^\alpha &= i\bar{\epsilon}_{gAB}\gamma^\alpha\chi_\mu^{AB},\\
\delta\chi_\mu^{AB} &= i\tilde{D}_\mu\epsilon_g^{AB},\\
\delta B_\mu{}^A{}_B &= i(\bar{f}^{\nu AC}\gamma_\mu\gamma_\nu\epsilon_{gBC} - \bar{f}^\nu_{BC}\gamma_\mu\gamma_\nu\epsilon_g^{AC})\\
&\quad+\tfrac{i}{4}(\bar{\epsilon}_{mBD}\gamma_\mu\Psi^{a(D}Z_a^{A)} - \bar{\epsilon}_m^{AD}\gamma_\mu\Psi_{a(D}\bar{Z}^a_{B)})\\
&\quad-\tfrac{i}{4}(\bar{\epsilon}_g^{AC}\chi_{\mu DC} - \bar{\epsilon}_{gDC}\chi_\mu^{AC})Z_a^D\bar{Z}^a_B - \tfrac{i}{4}(\bar{\epsilon}_g^{DC}\chi_{\mu BC} - \bar{\epsilon}_{gBC}\chi_\mu^{DC})Z_a^A\bar{Z}^a_D - trace\\
&\quad+\tfrac{i}{8}(\bar{\epsilon}_g^{AD}\chi_{\mu BD} - \bar{\epsilon}_{gBD}\chi_\mu^{AD})|Z|^2,\\
\delta Z_a^A &= i\bar{\epsilon}_m^{AB}\Psi_{Ba},\\
\delta\Psi_{Bd} &= \gamma^\mu\epsilon_{mAB}(\tilde{D}_\mu Z^A{}_d - i\hat{A}\bar{\chi}_\mu^{AD}\Psi_{Dd})\\
&\quad+ f^{ab}{}_{cd}Z^C{}_aZ^D{}_b\bar{Z}_B{}^c\epsilon_{mCD} - f^{ab}{}_{cd}Z^A{}_aZ^C{}_b\bar{Z}_C{}^c\epsilon_{mAB},\\
\delta A_\mu{}^a{}_b &= -i(\bar{\epsilon}_{mAB}\gamma_\mu\Psi^{Aa}Z_b^B - \bar{\epsilon}_m^{AB}\gamma_\mu\Psi_{Ab}\bar{Z}^a_B)\\
&\quad-2i(\bar{\epsilon}_g^{AD}\chi_{\mu BD} - \bar{\epsilon}_{gBD}\chi_\mu^{AD})Z_b^B\bar{Z}^a_A,
\end{aligned} \tag{5.34}
$$

provided the parameters introduced in this section are given the values

$$
\epsilon_g = \pm\tfrac{1}{\sqrt{2}}\epsilon_m,\;\; A = \pm\sqrt{2},\;\; A' = 1,\;\; A'' = \mp\sqrt{2},\;\; A''' = \tfrac{1}{2}. \tag{5.35}
$$

5.2 Comments on the results at this stage

We now turn to the hatted parameter $\hat{A}$. Having found the relation between the supersymmetry parameters in the ABJM and supergravity theories, we can also determine $\hat{A}$ by requiring that the variation of Ψ be supercovariant, which gives $\hat{A} = \pm\sqrt{2}$. This result will be confirmed in the next section. There we will also discover that supersymmetry does not demand that the $\chi\chi ZDZ$ term be supercovariantized

which is welcome because that would mean terms of the type $\chi^3\Psi Z$. In fact, from section two we know that the Lagrangian does not contain any terms at all with more than two explicit χ fields.

Recall also that we concluded that the R-symmetry gauge field $B_\mu{}^A{}_B$ is traceless when checking the local supersymmetry in the pure supergravity sector. This property must be implemented also after coupling it to matter. Inspecting the transformation rule of $\delta B_\mu{}^A{}_B$ above we see that this property is indeed satisfied also when the new terms are included.

Finally we would like to comment on the the abelian gauge field that is written out explicitly in the Lagrangian in section two. That this field provides an extra freedom at the order $(\tilde{D}_\mu)^2$ can be seen as follows. Introducing a charge q in the covariant derivative means that (suppressing R-symmetry indices)

$$[\tilde{D}_\mu, \tilde{D}_\nu]Z^a = \tilde{F}_{\mu\nu}{}^a{}_b Z^b + qF_{\mu\nu}Z^a. \tag{5.36}$$

The cancelation at this point in the analysis then works as follows. The relevant terms are

$$\delta L|_{D^2} = \epsilon^{\mu\nu\rho}(\tilde{F}_{\mu\nu}{}^a{}_b Z^b + qF_{\mu\nu}Z^a)J_{\rho a} + \epsilon^{\mu\nu\rho}(\delta A_\mu{}^a{}_b \tilde{F}_{\nu\rho}{}^b{}_a + \delta C_\mu F_{\nu\rho}), \tag{5.37}$$

which vanishes provided

$$\begin{aligned} \delta A_\mu{}^a{}_b &= -Z^a J_{\mu b}\,, \\ \delta C_\mu &= -qZ^a J_{\mu a}, \end{aligned} \tag{5.38}$$

that is, for any value of the charge q even though the structure of $J_{\mu a}$ is dictated by the theory. This fact will be made use of in the next section.

6. Cancelation of all terms in δL with one covariant derivative

In this section we continue the program of constructing a supersymmetric Lagrangian by considering the cancelation of all terms in δL that are of first order in the covariant derivative. As we will see below this will force us to introduce a number of new terms in the Lagrangian as well as to add further terms to the transformation rules presented at the end of the previous section.

Considering only the field content, there are six different kinds of terms in δL containing one derivative, two of these are bilinear in fermions, three are quartic and one is of sixth order in fermionic variables (including the supersymmetry parameter). Some structures come with different γ content and either with or without a structure constant which makes the list of independent terms to check a bit longer. We consider the cancelation of these terms in the order of increasing number of fermions. This will not fix the Lagrangian completely although the final form of the transformation rules will be determined. In the next section we will extend the analysis to terms in

δL which have two fermions and no derivatives. The information then obtained will be enough to give the final answer also for the Lagrangian.

To be more precise this part of the analysis will force us to add new terms to the supersymmetry transformation rules, that is, terms over and above those specified at the end of the last section. In particular we will need in $\delta\Psi$ new Z^3 terms without a structure constant:

$$\delta\Psi_{Bd}|_{new} = \tfrac{1}{4}Z^C_c Z^D_d \bar{Z}^c_B \epsilon_{CD} + \tfrac{1}{16}|Z|^2 Z^A_d \epsilon_{AB}. \tag{6.1}$$

It will also become clear from the calculations below that the underlying ABJM matter theory must be extended by an extra $U(1)$ gauge field as we have already mentioned in previous sections. In the final Lagrangian presented in section two the Chern-Simons term for this gauge field (denoted C_μ) was given explicitly. There its transformation rule was also presented and in this section we will see how these features of the theory arise.

Note that in the headings below $\cdot f$ refers to the fact that the term contains a structure constant $f^{ab}{}_{cd}$ and that the derivative can be acting on any of the fields although it is generally written as acting on the scalars.

6.1 Terms of second order in fermionic variables

6.1.1 $e(\bar{\epsilon}\gamma^\mu\Psi)\tilde{D}_\mu Z^3 \cdot f$

This calculation is needed already in the ungauged ABJM case. The new feature here is a remaining term where the derivative acts on the supersymmetry parameter. Such terms are easily canceled by adding new terms in the Lagrangian containing a χ_μ that when varied gives rise to the same kind of unwanted terms in δL but with opposite sign. These new terms are here of the form $e\chi\Psi Z^3 \cdot f$ and appear in the Lagrangian in section two as the terms on the line (2.6).

6.1.2 $e(\bar{\epsilon}\gamma^\mu\Psi)\tilde{D}_\mu Z^3$

The terms considered here are similar to the ones just analyzed apart from the important fact that they do not contain a structure constant. Such terms arise due to the presence of the new $\epsilon\Psi Z$ terms (without structure constants) that we found were necessary to add to δB_μ in the previous section. These new terms will, however, create problems when used in the variation of the Klein-Gordon term. Our approach to deal with this will contain additional modifications of the transformation rules together with new Yukawa-like terms without structure constants in the Lagrangian.

Thus we add to the Lagrangian the five possible structures that can be built out of two Ψ and two Z fields without using a structure constant. These terms are then varied under $\delta\Psi|_{DZ}$. To get this to work it turns out necessary to first modify $\delta\Psi$ by adding to it two Z^3 terms without structure constants (see (6.1)) and consider the variation of the Dirac term and, secondly, to introduce an extra $U(1)$ gauge field that

plays a special role. To this end we give the corresponding gauge field the following transformation rule

$$\delta C_\mu|_\psi = -iq(\bar{\epsilon}_{AB}\gamma_\mu\Psi^{Aa}Z^B_a - \bar{\epsilon}^{AB}\gamma_\mu\Psi_{Aa}\bar{Z}^a_B), \tag{6.2}$$

and the ABJM matter fields charge $q = \pm\frac{1}{4}$. When adding the corresponding variation of the Klein-Gordon term we find that there remains only a term with the derivative acting on the parameter. This last term we can cancel as usual by adding $\chi\Psi Z^3$ terms without structure constants to the Lagrangian.

6.1.3 $e\epsilon^{\mu\nu\rho}(\bar{\epsilon}\gamma_\mu\chi_\nu)\tilde{D}_\rho Z^4\cdot f$ and $e(\bar{\epsilon}\chi^\mu)\tilde{D}_\mu Z^4\cdot f$

These terms arise from the variation of the $\delta\tilde{A}_\mu$ in the Klein-Gordon term, the $\delta\Psi|_{Z^3\cdot f}$ variation in the first part of the supercurrent term and the $\delta\Psi|_{DZ}$ in the new $\chi\Psi Z^3\cdot f$ term in the Lagrangian. Adding these we find that the terms without $\epsilon^{\mu\nu\rho}$ cancel directly while the ones with an epsilon tensor do not. However, also the "f-terms" $\bar{f}\cdot\gamma\Psi Z$ varied under $\delta\Psi|_{Z^3\cdot f}$ contributes to the epsilon terms and when added leave only a $\tilde{D}_\mu\epsilon$ term which is canceled by adding an $\epsilon^{\mu\nu\rho}\bar{\chi}_\mu\gamma_\nu\chi_\rho Z^4$ term to L.

6.1.4 $e\epsilon^{\mu\nu\rho}(\bar{\epsilon}\gamma_\mu\chi_\nu)\tilde{D}_\rho Z^4$ and $e(\bar{\epsilon}\chi^\mu)\tilde{D}_\mu Z^4$

By varying the first part of the supercurrent under $\delta\Psi|_{Z^3}$ and the $\chi\Psi Z^3$ term under $\delta\Psi|_{DZ}$ we get contributions to both structures considered here. The epsilon tensor terms are canceled by the new $\delta\Psi|_{Z^3}$ variation (6.1) of the $\bar{f}\Psi Z$ term and the $\chi\chi Z^4$ terms without structure constants. To cancel the non-epsilon terms we need to vary the Klein-Gordon term with respect to B_μ to find that once again we seem to need a special $U(1)$ gauge field that varies into χ according to

$$\delta C_\mu|_\chi = -2iq(\bar{\epsilon}^{AD}_g\chi_{\mu BD} - \bar{\epsilon}_{gBD}\chi^{AD}_\mu)Z^B_a\bar{Z}^a_A\,, \tag{6.3}$$

again leading to the value $q = \pm\frac{1}{4}$. Note that to get the same value for q we have normalized this variation and the previous one $\delta C_\mu|_\Psi$ in the same way as for the corresponding terms in the variation of the non-abelian gauge field $\delta\tilde{A}_\mu$.

6.2 Terms quartic in fermionic variables

6.2.1 $e\bar{\epsilon}\chi\tilde{D}\Psi^2$

Terms of this kind come from the Dirac term by varying the dreibein, the super-covariant spin connection, the R-symmetry gauge field, and the ABJM spinor field itself. To these four contributions we add the terms obtained by performing a δZ variation in the first part of the supercurrent and in the so called f-term, namely $\bar{f}\cdot\gamma\Psi\bar{Z} + c.c.$. What remains to be canceled after these terms are added together are terms with the derivative acting on the susy parameter. These final terms are

exactly canceled by the variation of the second part of the supercurrent provided we write the ABJM Dirac term in a manifestly real way after gauging, i.e., by replacing

$$-i\bar{\Psi}^{Aa}\gamma^{\mu}D_{\mu}\Psi_{Aa} \rightarrow -\tfrac{ie}{2}\bar{\Psi}^{Aa}\gamma^{\mu}\tilde{D}_{\mu}\Psi_{Aa} - \tfrac{ie}{2}\bar{\Psi}_{Aa}\gamma^{\mu}\tilde{D}_{\mu}\Psi^{Aa}, \tag{6.4}$$

since this will mean that an otherwise required $\chi^2\Psi^2$ term is automatically accounted for.

6.2.2 $e\bar{\epsilon}\chi\chi\Psi\tilde{D}Z$

The analysis here fixes the coefficient in the Lagrangian of the term that would supercovariantize the $\chi\chi Z\tilde{D}Z$ term, namely $e\chi^3\Psi Z$. We find that this term has a vanishing coefficient and hence no term in the Lagrangian is of higher order than two in explicit χ fields.

The calculation goes as follows. We add the contributions from the δZ in $e\chi^2 Z\tilde{D}Z$, $\delta\Psi$ in the Dirac term and $e\chi^2\Psi^2$, δB_{μ} variations of the Chern-Simons term for the gravitino, δB_{μ}, $\delta\Psi$ and δe of the $\chi\Psi\tilde{D}Z$ term, the $\delta\omega_{\mu}$, δB_{μ}, $\delta\Psi$ and δe of the term $f\Psi Z$ and finally δZ of the $f\chi Z^2$ term. The result is

$$-\tfrac{1}{2}(\bar{f}^{\mu AB}\gamma^{\nu}\chi_{\mu AB})(\bar{\Psi}^{Ca}\gamma_{\nu}\epsilon_{CD})Z^D_a + c.c.\,. \tag{6.5}$$

However, this is exactly canceled by a term used already in the previous chapter on cancelation of $(\tilde{D}_{\mu})^2$ terms, namely the Riemann tensor term that arises in the $\delta\chi$ variation of $L_{A''}$. As we know from the supergravity analysis the double dual of the Riemann tensor is a second rank tensor whose symmetric piece is second order in derivatives while the antisymmetric part contains only one derivative. This latter tensor is just, after dualization, the triple dual whose variation gives the above term with opposite sign.

6.2.3 $e\bar{\epsilon}\chi\chi^2\tilde{D}|Z|^2$

Terms of this kind arise from the Chern-Simons term for the gravitino field, the $e\chi^2 Z\tilde{D}Z$ term, and the two terms $e\tilde{R}Z^2$ and $e\bar{f}\chi Z^2$. From the fact that these cancel we conclude that the term that would supercovariantize the $e\chi\chi Z\tilde{D}Z$ term, i.e. $e\chi^2 Z\chi\Psi$, has zero coefficient confirming the result obtained in the previous subsection. This calculation is similar to the one just above but makes instead use of the Riemann tensor coming from the variation of the term L_{RZ^2}.

6.3 Terms of sixth order in fermions

These terms are all of the form $e(\epsilon\chi)\tilde{D}\chi^4$ and do not arise explicitly in the variation of any of the terms in the Lagrangian. Thus all such terms are hidden in the covariant derivatives and therefore automatically dealt with when canceling the derivative terms.

6.4 Comments on the use of the $U(1)$ gauge field

Here we continue the discussion of the extra abelian vector field that was started at the end of the previous section. We saw there that it was possible to give the ABJM matter fields a charge q under the corresponding $U(1)$ gauge symmetry, and furthermore that this charge was not determined by the cancelation of terms of order $(\tilde{D}_\mu)^2$ in δL. However, as we have seen above, and now explain in more detail, the value of this charge is fixed by the order $\tilde{D}_\mu$ analysis performed in this section.

The terms relevant for this discussion are first of all the terms that remain after canceling the (non-ABJM gauge field) variations at first order in derivatives, that is,

$$\delta L|_{remaining} = (Z^a J_{\mu b})(K^\mu)^b{}_a + \tfrac{1}{16} Z^a J_{\mu a} (K^\mu)^a{}_a, \tag{6.6}$$

where $(K^\mu)^b{}_a$ is a fixed expression, and secondly the total matter gauge field variation

$$\delta L|_{\tilde{D}_\mu} = \delta \tilde{A}_\mu{}^a{}_b (K^\mu)^b{}_a + q\delta C_\mu (K^\mu)^a{}_a. \tag{6.7}$$

Combined with (5.38) these variations cancel each other provided $q^2 = \frac{1}{16}$. The corresponding new transformation law for the abelian gauge field is thus found to be

$$\begin{aligned} \delta C_\mu = &-iq(\bar{\epsilon}_{AB}\gamma_\mu \Psi^{Aa} Z^B_a - \bar{\epsilon}^{AB}\gamma_\mu \Psi_{Aa}\bar{Z}^a_B) \\ &-2iq(\bar{\epsilon}^{AD}_g \chi_{\mu BD} - \bar{\epsilon}_{gBD}\chi^{AD}_\mu) Z^B_a \bar{Z}^a_A. \end{aligned} \tag{6.8}$$

7. Cancelation of bifermion non-derivative terms in δL

In this section we add all terms in the Lagrangian that do not give any derivative contributions to its variation , i.e., different kinds of eZ^6 terms. By demanding cancelation of all non-derivative two-fermion terms in δL the coefficients in L of these eZ^6 terms are determined which finalizes the structure also of the Lagrangian.

The relevant terms that must cancel arise from the old and new Yukawa terms with Ψ varied into the ABJM Z^3 term with an f and the new terms of this kind without an f. Certain combinations of these expressions then cancel the contributions coming from varying Z in the various kinds of potential terms as will now be explained.

7.1 $e(\bar{\epsilon}\Psi)Z^5 \cdot f^2$

These f^2 terms are known to cancel already in the original ABJM computation, which is valid also here since no new contributions of this kind arise in the coupled theory.

7.2 $e(\bar{\epsilon}\Psi)Z^5 \cdot f$

These terms are similar to the previous ones but with only one structure constant. They arise from several sources: first from the $\delta\Psi|_{Z^3}$ variation of the ABJM Yukawa term and secondly from $\delta\Psi|_{Z^3 \cdot f}$ variation of the five new Yukawa like terms without structure constant. When adding these up the result can be seen to cancel the variation of Z in the new potential term with one f.

7.3 $e(\bar{\epsilon}\Psi)Z^5$

In the same fashion as for the previous cancelation these terms arise from the new Yukawa like terms without f by varying Ψ into Z^3 without f. Some of these terms eliminate each other, while the remaining terms cancel the variation of Z in the new f-free potential term.

7.4 $e(\bar{\epsilon}\gamma \cdot \chi)Z^6 \cdot f^2$

This kind of f^2 term comes from the $\delta\Psi|_{Z^3} \cdot f$ variation of the $\chi\Psi Z^3 \cdot f$ and must cancel the dreibein variation of the $Z^6 \cdot f^2$, i.e., the original ABJM potential term in L, which it does.

7.5 $e(\bar{\epsilon}\gamma \cdot \chi)Z^6 \cdot f$

Terms with one structure constant arise from the $\delta\Psi|_{Z^3}$ variation of $\chi\Psi Z^3 \cdot f$ in L and from $\delta\Psi|_{Z^3 \cdot f}$ variation of $\chi\Psi Z^3$. After cycling of the indices in one of the terms they cancel exactly the dreibein variation of the $Z^6 \cdot f$ term in L.

7.6 $e(\bar{\epsilon}\gamma \cdot \chi)Z^6$

These terms (without structure constant) arise from the $\delta\Psi|_{Z^3}$ variation of the $\chi\Psi Z^3$ terms in the action and from the variation of the dreibein in the Z^6 potential without structure constants. After cycling the $SU(4)$ indices in one of the terms, and using the self-duality relation, all terms can be seen to cancel.

8. Conclusions

In this paper we have coupled a general ABJM theory to the corresponding conformal supergravity theory constructed in section three of this paper. The proof of supersymmetry of the coupled theory has been carried through for all terms in δL with three, two and one derivative, together with all terms without derivatives that are bilinear in fermionic variables (including the susy parameter). This has been described in detail in the previous sections.

We will now discuss the remaining eight (non-derivative) terms in δL. Note that at this point in the analysis, i.e., before checking these last terms in δL, the

Lagrangian itself is in fact completely determined which is true also for the transformation rules. This follows from the fact that the only terms in the Lagrangian that do not generate any derivatives when varied are the pure eZ^6 terms. To determine their coefficients it is then sufficient to consider the cancelation of all terms in δL with two fermions. Concerning the transformation rules any term added at the non-derivative stage would alter parts of the previous calculations involving terms with derivatives and invalidate it.

Thus we conclude that the Lagrangian and the transformation rules presented in section two of this paper constitute the complete answer. The last terms in δL that must be analyzed in order to finalize the proof of supersymmetry are the following (non-derivative) ones, ordered in decreasing number of χ fields,

$$\bar{\epsilon}\chi\chi^6,\ \bar{\epsilon}\chi\chi^4 Z^2,\ \bar{\epsilon}\Psi\chi^4 Z,\ \bar{\epsilon}\chi\chi^2\Psi^2,\ \bar{\epsilon}\chi\chi^2 Z^4,\ \bar{\epsilon}\Psi\chi^2 Z^3,\ \bar{\epsilon}\chi\Psi^2 Z^2,\ \bar{\epsilon}\Psi\Psi^2 Z. \tag{8.1}$$

Of these the first one is part of the pure supergravity calculation, while the second and third are part of the covariant derivatives in the coupled theory since the torsion terms have been kept throughout the calculation. This fact also account for the fourth kind of term in the list. However, explicit terms with this field content arise in addition from varying, e.g., the dreibein in the Dirac term (plus an integration by parts) and from the term that supercovariantizes the supercurrent term in the Lagrangian. That the coefficient of this explicit $e\chi^2\Psi^2$ term in the Lagrangian is the correct one to provide this supercovarintization has been verified by checking the cancelation of terms in δL with one derivative. Of the remaining terms in the above list also the ABJM terms $e\bar{\epsilon}\Psi\Psi^2 Z$ have been verified to cancel. Thus the analysis includes in particular all terms in the original ABJM theory. What remains to be done is to check the cancelation of the fourth, fifth, sixth and seventh expressions in the list above. This is a rather elaborate calculation and has not yet been done in full detail.

Note that the last four structures in the list above can appear both with and without a structure constant f. Of these we have only, as just mentioned, checked the last one which is just an ABJM computation when it contains a structure constant. However, when it does not it is more interesting since then it makes use of the variation of the $U(1)$ gauge field C_μ in the Dirac term and therefore gives additional support for the way this field is being used here.

Since in this paper the parameters appearing in the Lagrangian and transformation rules are determined uniquely and in almost all cases from at least two separate calculations, we are fairly convinced that the cancelations that have not been established here will not alter any of our conclusions. Nevertheless, it would be welcome to find an independent argument for why the construction in this paper is correct. Methods that have been used in the past in similar circumstances are, e.g., constrained gauged superconformal algebras, superspace, the embedding tensor technique [15] and the construction of the on-shell supersymmetry algebra. Although

the first was used early on to obtain the pure conformal supergravity theories and the latter three were utilized in the more recent constructions of non-gravity M2 matter theories with eight (BLG), six (ABJM) or fewer supersymmetries, none of them seem to straightforwardly give an argument that would guarantee the existence of the type of coupled theories we are considering here. We hope to come back to these issues in a future publication.

It is worth remarking that the scalar potential after gauging contains, apart from the original ABJM terms with two structure constants, also terms with one as well as no structure constant (see the last two lines of the Lagrangian presented in section 2.1). As a further check of the derivation of these new contributions to the scalar potential one should verify that theory leads to an acceptable set of physical states. Another term that is crucial in this context is the conformal coupling between the curvature scalar and two scalar fields that arises in the process of checking supersymmetry. By giving the scalar field a vacuum expectation value the theory can be related to the corresponding one for a stack of D2 branes [16]. If we insert the VEV into the potential terms with one or two structure constants one finds that they do not contribute to the cosmological constant while the remaining potential terms (without structure constant) give a non-zero contribution. Using a VEV chosen such that it turns the $-\frac{e}{8}\tilde{R}|Z|^2$ term into a correctly normalized Einstein-Hilbert term, one finds a theory where this term is accompanied by a gravity Chern-Simons term and a cosmological constant. This part of the theory is described by the following Lagrangian

$$L = -\tfrac{e}{\kappa^2}(R + \tfrac{2}{\kappa^4}) + \tfrac{1}{2}\epsilon^{\mu\nu\rho}Tr(\omega_\mu\partial_\nu\omega_\rho + \tfrac{2}{3}\omega_\mu\omega_\nu\omega_\rho)\,. \tag{8.2}$$

We note that up to a sign this Lagrangian (with $\kappa^2 = 16\pi G$) is the same as the one of Li, Song and Strominger [17] at a chiral point[5]. The chirality is a welcome result while the sign may be problematic in view of the discussion in ref. [17] about the energy of physical states (black holes) and the central charge of the boundary CFT.

Some final comments are in order. First we note that the rather simple connection that exists between the $SU(2)\times SU(2)$ ABJM theory and the BLG theory seems less trivial after coupling these two theories to conformal supergravity. One complicating factor is that the topologically gauged ABJM theory seems to rely on the presence of an extra $U(1)$ gauge field. The supersymmetry exhancement of ABJM theories with abelian gauge fields has been discussed in [5, 6, 14]. It may also be of some interest to set the structure constants to zero eliminating the non-abelian parts of the ABJM gauge group and consider what might be a non-trivial new theory for one conformal M2 brane with six supersymmetries. A slightly more involved case arises if we set $f^{ab}{}_{cd} = \delta^{ab}_{cd}$ which also solves the fundamental identity.

[5]This remains the case also if a level (or dimensionless coupling constant) is introduced in the conformal gravity sector as discussed below. See [18] for further details.

In connection with the abelian gauge field and the charge $q = \pm\frac{1}{4}$ assigned to the matter fields, it may be interesting to reconsider the normalization of the Chern-Simons term since the level chosen for this term affects the value of q. In fact, since also gravitational Chern-Simons terms are associated with levels [19] the general issue of levels in topologically gauged BLG [7] and ABJM theories should be studied further. Note that if we introduce an independent level in the supergravity sector, or equivalently a dimensionless gravitational coupling constant at the classical level, it should appear in the Lagrangian and in the transformation rules in such a way as to make it possible to decouple the gravity and matter sectors by turning it off.

Acknowledgments

We would like to thank Ulf Gran, Horatiu Nastase and Andrew Strominger for discussions. The work is partly funded by the Swedish Research Council.

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